УДК 532.528

Asymptotic solutions and dependences for calculation of cavitation flows for slender axisymmetric bodies

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An attempt is made to elaborate different variants of asymptotic solutions to one of the problems, which are critical for practical applications, - determination of the shape of slender supercavities behind extended axisymmetric bodies. This study is aimed at development of simple engineering equations based on asymptotic solutions for the reliable shape estimation of cavities behind extended axisymmetric cavitators of quite arbitrary shapes.

Problems of reliable calculation, in case of cavities with very big elongations, appear to be very topical for a number of applications. However, considerable elongation of supercavities combined with a complex singular structure of solutions provides an essential hindrance for elaboration of quite reliable numerical solutions of these problems. Therefore, reliable nonlinear numerical solutions are currently available only for separate stationary test problems. At the same time, applications require quite simple methods of calculation, which are in some cases essentially distinct from test solutions. On the other hand, slenderness appeared to be a considerable simplifying factor, which allows one to construct quite simple asymptotic solutions. These solutions are very convenient as the basis for elaboration of simple and reliable methods of engineering calculations in the majority of applications.

Problem statement. The basics of hydrodynamics of supercavitation is presented in monography [1]. Within framework of the model of ideal incompressible liquid, the problem of determination of a supercavitating flow within the limits of hydrodynamics of slender bodies is reduced to the solution of the integro-differential equation for the cavity shape $r = R(x)$ behind a cavitator $r = r_n(x)$ [2, 3] for the initial conditions in the flow separation cross section and the condition for cavity length assessment (1) :

$$\frac{1}{2R^2}\left(\frac{dR^2}{dx}\right)^2 + \frac{d^2R^2}{dx^2}\ln\frac{R^2}{4x(L-x)} - \int_0^{x_s}\frac{\left.\frac{d^2r_n^2}{dx^2}\right|_{x=x_1} - \frac{d^2R^2}{dx^2}}{|x_1-x|}dx_1 -$$

$$(\ln 1/\delta_*^2)^{-1} \qquad (1) \qquad (\ln 1/\delta_*^2)^{-1}$$

$$-\int_{x_s}^{L}\frac{\left.\frac{d^2R^2}{dx^2}\right|_{x=x_1} - \frac{d^2R^2}{dx^2}}{|x_1-x|}dx_1 - \frac{\left.\frac{dr_n^2}{dx}\right|_{x=0}}{x} + \frac{\left.\frac{dR^2}{dx}\right|_{x=L}}{L-x} = 2\sigma(x) \qquad (1)$$

$$(\ln 1/\delta_*^2)^{-1} \qquad (\ln 1/\delta_*^2)^{-1} \qquad (\ln 1/\delta_*^2)^{-1} \qquad (1)$$

$$\left[R = r_n(x)\right]_{x=x_s}, \left[\frac{dR^2}{dx} = \frac{dr_n^2}{dx}\right]_{x=x_s} \left[R^2 = 0\right]_{x=L}$$

Here $x = x_s$ is flow separation cross section coordinate, $\sigma = 2(P_\infty - P_c)/\rho U_\infty^2$ is the cavitation number, where $P_\infty - P_c$ is the difference of hydrostatical pressure and pressure in a cavity, ρ is mass density of the liquid, and U_∞ is inflow velocity. The parameter $\delta_* = 2R_m / l + L_c$ defines the order of magnitude of the ratio between the maximal diameter of a cavity $2R_m$ and the total length $l+L_c$ of the cavitator - cavity system and is assumed to be a small parameter. Below each term of equation (1) there is indication of the order of its smallness at $\delta_* \to 0$.

Regular solutions. *The direct solution method for preset cavitator length:* The Asymptotic solution of problem (1) is obtained for the preset values of: the length *l*=1 of quite arbitrarily shaped cavitator $r_n = r_n(x) = \delta \tilde{r}_n(x)$ and cavitation number $\hat{\sigma}(x) = O(1)$ under condition that $\varepsilon / \delta = O(1)$ and for the preset number of various alternatives $\delta = O(\varepsilon)$. In case of a cone, $\varepsilon = \tan\gamma$, where γ is the cone semi-opening angle. The flow separation cross-section is assumed to be fixed (Fig. 1). The solution for the cavity shape and length L_c is derived in the form of expansions:

$$R^2 = \delta^2\left[\tilde{R}_o^2 + \frac{1}{\ln(1/\delta^2)}\tilde{R}_1^2 + \dots\right],\; L_c = \tilde{L}_{co} + \frac{\tilde{L}_{c1}}{\ln(1/\delta^2)} + \dots \quad (2)$$

and is further reduced to the sequence of boundary problems

$$\frac{d^2\tilde{R}_o^2}{dx^2} = -2\hat{\sigma}(x) \quad \left[\frac{d\tilde{R}_o^2}{dx} = \frac{d\tilde{r}_n^2}{dx}\right]_{x=0}, \quad \left[\tilde{R}_o^2 = \tilde{r}_n^2\right]_{x=0};$$

$$\frac{d^2\tilde{R}_1^2}{dx^2} = \frac{1}{2\tilde{R}_o^2}\left(\frac{d\tilde{R}_o^2}{dx}\right)^2 + \frac{d^2\tilde{R}_o^2}{dx^2}\ln\left(\frac{\tilde{R}_o^2}{4(1+x)(1-L_{co})}\right) - \int_{-1}^{0}\frac{\left.\frac{d^2\tilde{r}_n^2}{dx^2}\right|_{x=x_1} - \frac{d^2\tilde{R}_o^2}{dx^2}}{|x_1 - x|}dx_1 - \quad (3)$$

$$-\int_0^{L_{co}}\frac{\left.\frac{d^2\tilde{R}_o^2}{dx^2}\right|_{x=x_1} - \frac{d^2\tilde{R}_o^2}{dx^2}}{|x_1 - x|}dx_1 - \frac{\frac{d\tilde{r}_n^2}{dx}\Big|_{x=-1}}{1+x} + \frac{\left.\frac{d\tilde{R}_o^2}{dx}\right|_{x=L_o}}{L_{co} - x}, \quad \left.\frac{d\tilde{R}_1^2}{dx}\right|_{x=0} = 0,\; \left.\tilde{R}_1^2\right|_{x=0} = 0$$

The solution in the form of two terms of a series is generally derived in quadratures:

$$R^2 = R_o^2 + \frac{1}{\ln(1/\delta^2)}R_1^2 = [\varepsilon^2 + 2m\varepsilon^2 x] - \left[\int_0^x \frac{(x-x_1)2\sigma(x_1)}{\ln(1/\delta^2)}dx_1\right] + \frac{1}{\ln(1/\delta^2)}\left[\int_0^x (x-x_1)\frac{d^2R_1^2}{dx_1^2}dx_1\right] \quad (4)$$

The solution for a cone *m* =1 is derived in an analytical form [3], but calculation on the basis of solution (4) is more convenient. Fig. 1 illustrates accuracy of the solution (4) for a cone with σ=0.04, γ=10°, and *l* = 1 in comparison with the nonlinear numerical calculation [5].

Semi-inverse solution method. Equations of problem (1) are written in the coordinate system with *x*=0 in the cavity mid-section. The solution is derived for the preset length L_e =1 of the cavity rear part, which adjoins its mid-section by the slenderness parameter of this part of cavity, provided the condition $\varepsilon / \delta = O(1)$ is satisfied, in the form of expansions similar to (2) for initial conditions (5a):

$$\text{a)}\; R^2(x)\Big|_{x=0} = \delta^2,\; \left.\frac{dR^2(x)}{dx}\right|_{x=0} = 0,\; R^2(x)\Big|_{x=1} = 0,\quad \text{b)}\; \left.\frac{dR^2}{dx}\right|_{x=L_m} = 2R_n\tan\gamma,\quad R^2\Big|_{x=L_m} = R_n^2, \quad (5)$$

where $\delta = 1/\lambda = R_m / L_e$, R_m being the maximal radius of a cavity. Conditions (5b) are thus set, which control the cavity radius R_n, the inclination angle of its meridian γ in the flow separation cross section, as well as location of the flow separation cross section $x=L_m$. When the problem solution was derived, it became possible to find dependence (6a) between the cavitation number and parameter λ of its rear part, in case of a cavity behind a cone.

$$\text{a) } \sigma = \frac{\ln \lambda^2 / \beta^2}{\lambda^2}\left[1 - 2\left(1 + \varepsilon\lambda\sqrt{1+(\varepsilon\lambda)^2}\ \ln\frac{\varepsilon\lambda}{\sqrt{1+(\varepsilon\lambda)^2}}\right)\frac{1}{\ln \lambda^2 / \beta^2}\right],\ \varepsilon = \tan\gamma: \ \text{b) } \sigma = \frac{2}{\lambda^2}\ln\frac{\lambda}{e}, \quad (6)$$

Expression (6a) turned out to be more universal, yielding in the limiting cases the dependence for λ of a cavity behind a cylinder (6b) and the earlier derived dependence from elongation, in case of a small disk-type cavitator (6c) [3]. Results of calculations via dependence (6a) are depicted in Fig. 2. For the same cavitation numbers, according to dependences (6b, 6c), a cavity behind a cylinder appears essentially shorter than the rear part of a cavity behind a disk.

Engineering method of calculation of cavities behind slender axisymmetric cavitators. The refined variant of the system of equations [2], which is suitable for calculation of cavities behind slender cavitators, has the following form:

$$\text{a) } \mu_c \frac{d^2R^2}{dx^2} + \sigma(x) = 0, \quad \text{b) } \left.\frac{dR^2}{dx}\right|_{x=0} = R_n\sqrt{\frac{2(c_d - k\sigma)}{k\mu_c}}, \quad \text{c) } \left.R^2\right|_{x=0} = R_n^2, \quad (7)$$

The basic idea of deriving of these equations implies that the solution for the cavity shape is based on the differential equation (3), which is the first approximation of the integro-differential equation (1). However, the characteristic factors available in the problem solution are defined on the basis of solving more exact integro-differential equation (1). The main advantage of equations (7), in addition to their simplicity, is the exclusive universality of their applicability for calculations that imply both conditions $\varepsilon / \delta = O(1)$, and $\varepsilon / \delta \to 0$ for quite arbitrary shapes of cavitators and dependences from the cavitation number. A very weak dependence of factors in these equations from deformation of cavitators and cavities is observed, which is confirmed experimentally by the well-known principle of independence of expansion of a cavity [1]. This allows one to use values factors derived for the basic form of a cavitator and a cavity in calculations of other forms, which are not too distinct from the basic one. Equations (7) contain two characteristic parameters μ and k, which have a clear physical meaning. Value μ characterizes the inertia properties of the expanding cavity cross section, being a certain inertia factor. This value in the form of dependence (8b) is asessed on the basis of the second- order solution for cavity elongation λ and the second approximation of the problem solution (8a) for the case of $\delta / \varepsilon \to 0$ [3, 4]. The same dependence in form (8g) is also obtained at $\lambda \to \infty$ for regular $\delta / \varepsilon = 0(1)$ asymptotic solution (6a).

$$\text{a) } \sigma = \frac{2}{\lambda^2}\ln\frac{\lambda}{\sqrt{e}} = \frac{2\mu}{\lambda^2}, \quad \text{b) } \mu = \ln\frac{\lambda}{\sqrt{e}} \to \quad \text{c) } \mu \approx \ln\sqrt{\frac{\ln 2/\sigma}{e\sigma}},$$

$$\text{d) } \mu_c = \mu\frac{0.82}{(0.82 - k\sigma)}; \quad \text{e) } \mu = \frac{\ln\lambda^2}{\lambda^2}\left[1 - 2\left(1 + \varepsilon\lambda\sqrt{1+\varepsilon^2\lambda^2}\ \ln\frac{\varepsilon\lambda}{\sqrt{1+\varepsilon^2\lambda^2}}\right)\frac{1}{\ln\lambda^2}\right], \quad (9)$$

$$\text{f) } \mu_{\varepsilon\to 0} \to \ln\frac{\lambda}{e}, \quad \text{g) } \mu_{\lambda\to\infty} \to \ln\frac{\lambda}{\sqrt{e}}$$

Results of calculation of factors μ depending on λ for a cavity behind a cone with $\gamma = 10^{o}$ (8e) and a cylinder (8f) at $\beta = 1$ are depicted in Fig. 3.

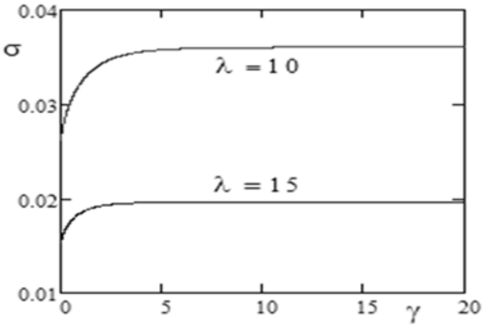

Fig. 2. Cavitation number σ in dependence (6a) from cone semi-opening angle γ for the fixed elongation values λ=10 and λ=15 of the rear part of a cavity behind its mid-section.

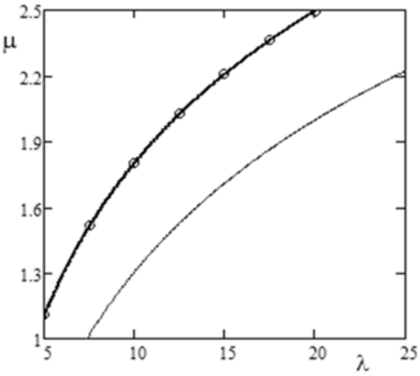

Fig.3. Dependence (8e) for the inertia factor $\mu = \mu(\gamma, \lambda)$, $\beta = 1$, in case of a cavity behind a cone ▬▬ a cone $\gamma = 10^o$, —— a cylinder ○ ○ ○ ○ Numerical calculation $\mu = \sigma\lambda^2$, [5]

As the first initial condition (7b) of the problem (7), the equation of conservation of energy transferred from cavitator to cavity cross sections at the initial moment is used. This condition is the equivalent of merging and it simulates a small intermediate area in the cavitator's vicinity by a jump in the cavity meredian inclination angle. Value $k \sim 0.93 - 1$ in dependence (9a) characterizes a small-scale longitudinal transfer of energy along cavity cross sections. This value is calculated on the basis of the second approximation of the solution at $\delta / \varepsilon \to 0$. Correction (8d) is applied for not small enough values of σ.

A formal solution (10) at σ=const of the first approximation of the regular problem of type (2-4), in case of equality of the inclination angles of cavitator and cavity meridians in the form of an ellipsoidal cavity is obtained as follows:

$$R^2 = R_n^2 + \left(\frac{dr_n^2}{dx} \bigg|_{x=0} \right) x - \frac{\sigma}{\ln(1/\delta^2)} x^2 \tag{10}$$

At σ=const the equation for engineering calculation (7a) also controls an ellipsoidal cavity (11) which, however, at the corrected inclination angle of cavity meridian in the flow separation cross section takes the following form:

$$\text{a) } R^2 = R_n^2 + R_n \sqrt{\frac{2(c_d - k\sigma)}{k\mu_c}} x - \frac{\sigma}{2\mu_c} x^2 \quad \text{b) } R_m = R_n \sqrt{\frac{c_d}{k\sigma}}, \quad \text{b) } L_c = \frac{R_n}{\sigma} \sqrt{\frac{2\mu_c}{k}} \left(\sqrt{c_d - k\sigma} + \sqrt{c_d} \right) \tag{11}$$

This equation allows one to derive dependences for the cavity maximal radius R_m (11b) and length L_c (11c). Results of calculation of the solution (11a) for a cavity behind a cone at $\gamma = 10^o$, $\sigma = 0.04, 0.02$, $R_n = 1$ $\gamma = 10^o$ $\sigma = 0.04, 0.02$, $R_n = 1$, in comparison with the results of nonlinear numerical calculation and experimental date[5-7], are depicted in Fig. 4.

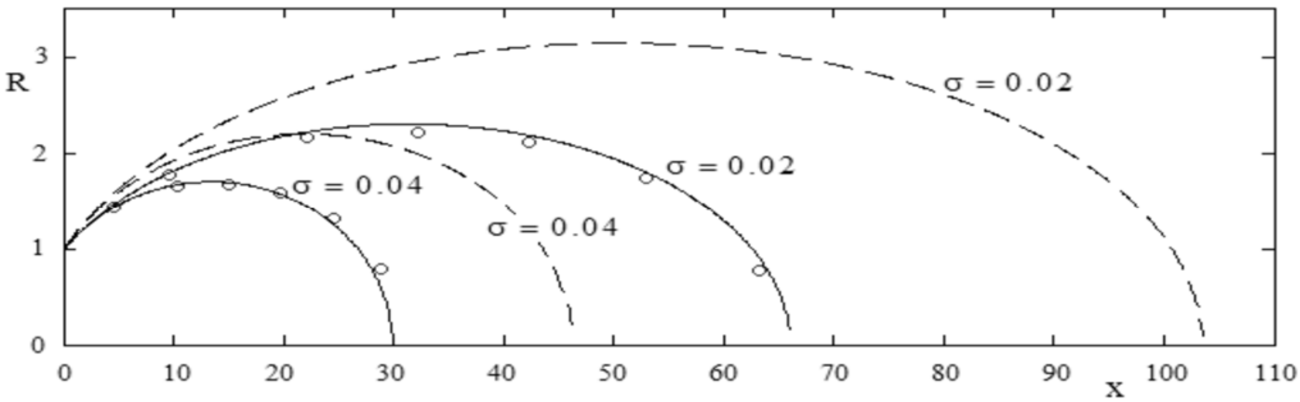

Fig. 4. Results of calculation for a cavity behind a cone $\gamma = 10^o$ $\sigma = 0.04, 0.02$, $R_n = 1$ on the basis of the equations for engineering calculation (7).

— — Solution (11a) on the basis of the equations for engineering calculation (7)

○ ○ ○ ○ Nonlinear numerical calculation [5]

——— The formal solution of the first approximation (10) in the form of an ellipsoidal cavity by parameter $\delta_* = 2R_m / (l + L_c)$ in case of equality of the inclination angles of cavitator and cavity meridians in the flow separation cross section.

Conclusions:

- Calculations of cavities formed behind extended cavitators of quite arbitrary shapes can be conducted using simple engineering equations.

Reverences

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